Discrete Applied Mathematics 8 (1984) 69-83
North-Holland

CHRONOLOGICAL ORDERINGS OF INTERVAL GRAPHS

D. SKRIEN
Department of Mathematics, Colby College, Waterville, ME 04901, USA

Received 18 June 1980
Revised 16 April 1982 and 31 March 1983

If an undirected graph is the intersection graph of a set of intervals of the real line, it is called an *interval graph* and the set of intervals is called an *interval representation* of the graph. An interval graph typically has many representations that differ in the order of the endpoints of the intervals along the line. This paper gives three methods for describing these differences and shows how these methods can be used to determine whether a graph has a representation satisfying various restrictions on the relative positions of the intervals. It concludes with an application of these results to the subject of interval counts of interval graphs.

1. Introduction

Let an undirected graph $G=(V,E)$ have vertex set $V=\{v_1,v_2,\ldots,v_n\}$. The graph G is called an *interval graph* if there exists a set $\{I_1,\ldots,I_n\}$ of intervals of the real line such that, for $i\neq j$,

$$\{v_i,v_j\}\in E \quad \text{iff} \quad I_i\cap I_j\neq\emptyset.$$

The set $\{I_1,\ldots,I_n\}$ is called an *interval representation* of G. We will assume, without loss of generality, that all interval representations consist of closed, nonempty intervals.

Interval graphs arise naturally in archaeological seriation problems (see [8], [9], [11], [12]). In studying the artifacts obtained from ancient graveyards, archaeologists may wish to determine the intervals of time in which the various styles of pottery or other artifacts were in use. As a first step, they can often determine which styles had overlapping periods of use by noting which styles appear in common graves. Assuming each style was in use in only one interval of time, and assuming the archaeologists can correctly determine which intervals of use overlapped, we can construct an interval graph $G=(V,E)$ as follows: Let $V=\{v_1,\ldots,v_n\}$ be the set of styles of pottery and connect v_i to v_j with an edge if and only if the interval of use of style v_i overlapped the interval of use of style v_j. The various interval representations of G then become the possible chronological representations for the styles of pottery.

However, G typically has many different interval representations, differing not only in the lengths of the intervals, but more importantly in the relative positions of the intervals. It would be useful to the archaeologists to know which of these representations corresponds to the styles of pottery under study. Furthermore, the archaeologists may have already determined certain relationships among the intervals of use of the various styles which he would like incorporated in his interval representation. For example, he might know that style v_i disappeared before style v_j disappeared. Then he would be interested only in those representations of his interval graph in which interval I_j extends to the right of interval I_i.

This application of interval graphs to archaeological seriation provides a motivation for the subject matter of this paper. We study the possible relative positions of the intervals in the representations of an interval graph.

2. Chronological orderings

Let $G=(V,E)$ be an interval graph with n vertices, and let $\{I_1,\dots,I_n\}$ denote an interval representation of G in which the endpoints of the intervals are all distinct. Let P denote a reference set of $2n$ elements $\{l_1,\dots,l_n,r_1,\dots,r_n\}$. If we now associate the left [resp., right] endpoint of interval I_i with l_i [resp., r_i] from P, for $i=1,\dots,n$, then the linear order of the endpoints of the intervals along the real line induces a linear ordering of the elements of P.

It is useful to look at linear orderings of the elements of P as transitive orientations of a graph. Consider the set P as a set of vertices, and connect each pair of them with an edge to form a complete undirected graph $Q(G)$. We can *orient* this graph (or the edges of the graph) by assigning a direction to each edge to form a digraph $D=(P,T)$. The arcs of D are denoted by ordered pairs of distinct vertices. The set T of arcs is called an *orientation* of $Q(G)$ (or of the edges of $Q(G)$). If T is also *transitive*, i.e., if $(a,b),(b,c)\in T$ implies that $(a,c)\in T$, then T induces a linear ordering of P. Therefore our questions about linear orderings of P will often be translated into questions about transitive orientations of $Q(G)$, and vice versa.

We wish to study those linear orderings of P induced by an interval representation of an interval graph G. We call such linear orderings of P *chronological orderings* of G.

Given an interval graph $G=(V,E)$, it is easy to determine which of the $(2n)!$ possible linear orderings of P actually are chronological orderings of G. They are exactly those linear orderings T with the following property:

$$(l_j,r_i),(l_i,r_j)\in T \quad \text{iff} \quad i=j \text{ or } \{v_i,v_j\}\in E.$$

In fact, linear orderings of P with this property exist iff G is an interval graph.

We wish to determine a method of finding chronological orderings that satisfy various given restrictions on the orders of the elements. This can be more precisely stated:

Given an interval graph $G=(V,E)$ and a partial orientation S of $Q(G)$, does there exist a transitive orientation of $Q(G)$ that extends S and is a chronological ordering of G? If so, find such an extension.

Theorem 1 below provides an answer to this question. Furthermore, it provides an algorithm for constructing the desired extension of S, if such an extension exists. We first recall the theorem in [4] which states that G is an interval graph iff G has no induced cycles of length 4 and $\bar{G}$ (the complementary graph of G) is transitively orientable. Furthermore, if this is the case, then for each transitive orientation O of $\bar{G}$, there exists a corresponding interval representation $\{I_1, \ldots, I_n\}$ of G in which $(v_i, v_j) \in O$ iff $I_i < I_j$, where $I_i < I_j$ means interval I_i is completely to the left of interval I_j.

Theorem 1. *Let G be an interval graph and let S be a partial orientation of $Q(G)$. S can be extended to a transitive orientation of $Q(G)$ that is a chronological ordering of G iff the following three conditions hold:*

(a) *S is acyclic.*

(b) *For all i and j such that $i = j$ or $\{v_i, v_j\} \in E$, $(r_i, l_j) \notin S$.*

(c) *There exists a transitive orientation O of $\bar{G}$ with the following properties:*

(i) *For all $(x_i, y_j) \in S$ such that $i \neq j$ and $\{v_i, v_j\} \notin E$, we have $(v_i, v_j) \in O$. (Here x and y are either r or l.)*

(ii) *For all $(r_i, r_j) \in S$ such that $\{v_i, v_j\} \in E$ and for all $k \neq j$ such that $\{v_i, v_k\} \in E$ but $\{v_j, v_k\} \notin E$, we have $(v_k, v_j) \in O$.*

(iii) *For all $(l_i, l_j) \in S$ such that $\{v_i, v_j\} \in E$ and for all $k \neq i$ such that $\{v_j, v_k\} \in E$ but $\{v_i, v_k\} \notin E$, we have $(v_i, v_k) \in O$.*

Proof. Conditions (a) and (b) are clearly necessary.

Any chronological ordering T of G that extends S gives an interval representation of G that in turn induces a transitive orientation O of $\bar{G}$, as discussed prior to the theorem. Now, T and O must be related as described in parts (i), (ii), and (iii) of condition (c), as one can easily see by considering the interval representation of G obtained from T. (For parts (ii) and (iii), see Fig. 1.) Therefore condition (c) is also necessary.

The proof of sufficiency is more difficult. We proceed by constructing the desired extension of S. We first use O to construct a partial orientation W of $Q(G)$ that contains all the orientations necessary and sufficient for a linear ordering of P to be a chronological ordering of G. We then show that $W \cup S$ produces an acyclic partial orientation of $Q(G)$ which therefore can be extended arbitrarily (but consistently) to obtain a linear ordering of P as desired.

We first use O to construct a partial orientation W of $Q(G)$ as follows:

(1) For all $\{l_i, r_j\}$ such that $i = j$ or $\{v_i, v_j\} \in E$, place (l_i, r_j) in W.

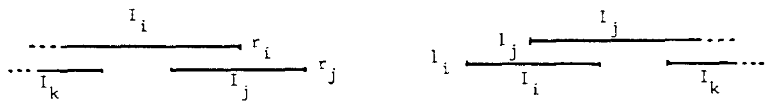


Fig. 1. The necessity of parts (ii) and (iii) in condition (c) of Theorem 1.

(2) For all $(v_i, v_j) \in O$, place $(r_i, l_j), (r_i, r_j), (l_i, r_j)$, and (l_i, l_j) in W.

(3) For all $(v_i, v_k) \in O$, and for all j such that $\{v_i, v_j\}, \{v_j, v_k\} \in E$, place (r_i, r_j) and (l_j, l_k) in W.

We first need to show that W is a well-defined orientation, i.e., that no edge of $Q(G)$ has been oriented in both directions in W. However, G is an interval graph, so it has an interval representation corresponding to each transitive orientation of $\bar{G}$. In particular, it has a representation corresponding to our orientation O of $\bar{G}$. From this representation we can obtain a chronological ordering of G. But it is easy to see that any chronological ordering of G obtained from a representation that uses O must contain W. This means that W must be well-defined.

To complete the proof, we need to show that $W \cup S$ produces an acyclic partial orientation of $Q(G)$. We first note that, by the definition of W and conditions (b) and (c), $W \cup S$ is a well-defined partial orientation of $Q(G)$ with no edges oriented in both directions. To show that $W \cup S$ is acyclic, we prove that W is transitive and that the graph whose vertices are P and whose edges are those edges of $Q(G)$ *not* oriented by W only contains components consisting of cliques. When combined with the acyclicity of S, this yields the acyclicity of $W \cup S$.

We proceed to show that W is transitive. First, note that, since W is contained in at least one linear order of $Q(G)$, it must be acyclic. Due to this fact, due to the fact that parts (1) and (2) in the definition of W cause every edge of the form $\{r_i, l_j\}$ in $Q(G)$ to be oriented in W, and due to symmetry, we need only prove the following two statements:

(s_1) If $(r_i, r_j), (r_j, r_k) \in W$, then $(r_i, r_k) \in W$, and

(s_2) if $(r_i, l_j), (l_j, r_k) \in W$, then $(r_i, r_k) \in W$.

To prove (s_1), let $(r_i, r_j), (r_j, r_k) \in W$. Assume first that $\{v_i, v_k\} \notin E$. Then either $(v_i, v_k) \in O$ or $(v_k, v_i) \in O$, and so we must have either $(r_i, r_k) \in W$ or $(r_k, r_i) \in W$ by the definition of W. Since W is acyclic, we must have $(r_i, r_k) \in W$, which gives us (s_1).

Consider now the case where $\{v_i, v_k\} \in E$. We use the following notation in the remainder of the proof. By part (3) of the definition of W, (r_i, r_j) and (l_j, l_k) are added to W whenever $\{v_i, v_j\}, \{v_j, v_k\} \in E$ and $(v_i, v_k) \in O$. We indicate this by saying that the triple $[v_i, v_j, v_k]$ *forces* (r_i, r_j) and (l_j, l_k) into W.

If $\{v_i, v_j\} \notin E$, but $\{v_j, v_k\} \in E$, then by the definition of W, $(v_i, v_j) \in O$ and so $[v_i, v_k, v_j]$ forces $(r_i, r_k) \in W$ as desired.

If $\{v_i, v_j\} \in E$, but $\{v_j, v_k\} \notin E$, then $(v_j, v_k) \in O$ and so $[v_j, v_i, v_k]$ forces $(r_j, r_i) \in W$, contradicting the fact that $(r_i, r_j) \in W$. Therefore this situation cannot occur.

If $\{v_i, v_j\}, \{v_j, v_k\} \notin E$, then the transitivity of O is contradicted, because, in that case, we must have $(v_i, v_j), (v_j, v_k) \in O$, whereas $\{v_i, v_k\} \in E$.

Now assume that $\{v_i, v_j\}, \{v_j, v_k\} \in E$. Then due to the fact that $(r_i, r_j) \in W$, there must exist some v_l such that $(v_i, v_l) \in O$ and $\{v_j, v_l\} \in E$ (see Fig. 2) and such that $[v_i, v_j, v_l]$ forces $(r_i, r_j) \in W$. Consider the edge $\{v_k, v_l\}$. If $\{v_k, v_l\} \notin E$, then by the transitivity of O, $(v_k, v_l) \in O$ and therefore $[v_k, v_j, v_l]$ forces $(r_k, r_j) \in W$, contradict-

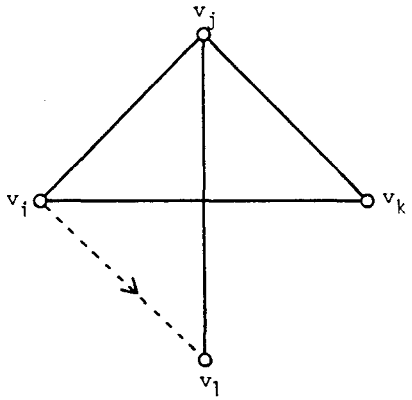


Fig. 2.

ing our original information. Hence we must have $\{v_k, v_l\} \in E$. But now, we see that $[v_i, v_k, v_l]$ forces $(r_i, r_k) \in W$, which proves statement (s_1).

We now prove statement (s_2). As in the proof of statement (s_1), if $\{v_i, v_k\} \notin E$, then $(r_i, r_k) \in W$ as desired, so let us assume that $\{v_i, v_k\} \in E$. Note that we must have $\{v_i, v_j\} \notin E$. More specifically, $(v_i, v_j) \in O$ by the definition of W. Hence, in particular, $j \neq k$. Now if $\{v_j, v_k\} \notin E$, then $(v_j, v_k) \in O$, since $(l_j, r_k) \in W$. However, this contradicts the transitivity of O. Therefore, we must have $\{v_j, v_k\} \in E$. But now $[v_i, v_k, v_j]$ forces $(r_i, r_k) \in W$, as desired. This completes the proof of the transitivity of W.

For the remainder of the proof, let B denote the edges of $Q(G)$ not oriented in W. We show that the components of the graph (P, B) are cliques. As remarked earlier, all the edges of the form $\{l_i, r_j\}$ are oriented in W. Therefore, by symmetry, we need only prove that if $\{r_i, r_j\}, \{r_j, r_k\} \in B$, then $\{r_i, r_k\} \in B$. To prove this, we first note that $\{v_i, v_j\}, \{v_j, v_k\} \in E$, for otherwise, part (2) in the definition of W would have oriented at least one of $\{r_i, r_j\}$ or $\{r_j, r_k\}$. Therefore $\{v_i, v_k\} \in E$ as well, because otherwise we would have $(v_i, v_k) \in O$ or $(v_k, v_i) \in O$, and then $[v_i, v_j, v_k]$ forces $(r_i, r_j) \in W$ or $[v_k, v_j, v_i]$ forces $(r_k, r_j) \in W$, contradicting our assumptions that $\{r_i, r_j\}, \{r_j, r_k\} \in B$. We complete the proof by assuming that $\{r_i, r_k\} \notin B$ and getting a contradiction. Without loss of generality, let $(r_i, r_k) \in W$. For this to occur, there must be a vertex v_l with $(v_i, v_l) \in O$ and $\{v_k, v_l\} \in E$ (see Fig. 3) from which $[v_i, v_k, v_l]$ could force $(r_i, r_k) \in W$. But now consider $\{v_j, v_l\}$. If $\{v_j, v_l\} \in E$, then $[v_i, v_j, v_l]$ forces $(r_i, r_j) \in W$, contradicting our assumption that $\{r_i, r_j\} \in B$. If $\{v_j, v_l\} \notin E$, then the transitivity of O forces $(v_j, v_l) \in O$ and so now $[v_j, v_k, v_l]$ forces $(r_j, r_k) \in W$, again giving us a contradiction. Thus we must have $\{r_i, r_k\} \in B$. This proves that the components of (P, B) are cliques.

Now, from the transitivity of W and the fact that the components of (P, B) are cliques, it is easy to see that $S \cup W$ is acyclic. Therefore, by extending this orientation arbitrarily to a linear ordering of $Q(G)$, we obtain the desired extension of S. This completes the proof of the theorem. $\square$

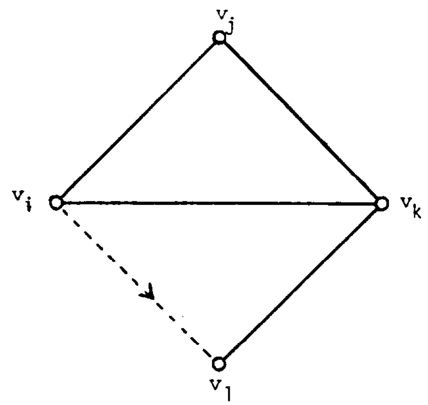


Fig. 3.

This theorem gives us an algorithm for determining whether S has the desired extension and for constructing such an extension if it exists. One can check properties (a) and (b) in Theorem 1 in time $O(n+|S|)$. To check property (c), we can first use (i), (ii), and (iii) to partially orient $\bar{G}$, which takes $O(|S|+ne)$ steps, where e is the number of edges in G. We then extend this (if possible) to a transitive orientation of $\bar{G}$. The problem of extending partial orientations of a graph to a transitive orientation of the graph is discussed in [3]. One can also use the methods in [5], [6], [7], whose algorithm enables us to obtain (if possible) such an orientation of $\bar{G}$ in $O((n-d)(n^2-e))$ steps, where d is the minimum degree of the vertices of G. Then one can construct the orientation W in $O(n^2+ne)$ steps. Adding on the orientations from S, and then extending this to a linear ordering of $Q(G)$ using a linear-time topological sorting algorithm takes $O(n^2)$ steps (because $S \cup W$ will contain at least n^2 arcs). The overall efficiency of our algorithm is $O(ne+n^2+(n-d)(n^2-e))$, which reduces to $O(n^3)$.

One can note that the theorem can be made to hold for any graph G instead of just interval graphs by including the extra condition that (d) G is triangulated, or (d') G has no induced cycle of length four. This can be checked in $O(n+e)$ time.

We now present a slight variation of the problem presented above. In our motivational discussion of the archaeological seriation problem we assumed that the archaeologist could determine exactly which intervals of use overlapped. This is somewhat unrealistic. In attempting to construct the interval graph for the artifacts, the data suggesting the inclusion (or omission) of certain edges might be insufficiently compelling. (A similar problem was encountered by the geneticist Benzer [1], [2] in testing the intersections of 145 regions of a chromosome. He was not able to determine whether or not every pair of regions intersected.) In this case, one has the problem of attempting to construct an interval graph when the existence of some of its edges is uncertain.

In our seriation application, the problem can be solved under certain assumptions about the available information. For example, consider two styles that the archaeologists are certain had non-overlapping intervals of use. Then it is also reasonable to assume that they know which style is older and which is newer. Under this assumption, our problem can be formulated as follows: Given a graph $G=(V,E)$ whose edges are partitioned into two sets, E_1 (where we are certain of overlap) and E_2 (where we are uncertain of overlap), and an orientation O of $\bar{G}$, does there exist a set of intervals $\{I_1, \dots, I_n\}$ such that

(a) if $\{v_i, v_j\} \in E_1$, then $I_i \cap I_j \neq \emptyset$ and

(b) if $(v_i, v_j) \in O$, then $I_i < I_j$?

This question can be easily answered. One need only form a partial orientation of $Q(G)$ as follows:

(1) For all i, j such that $i=j$ or $\{v_i, v_j\} \in E_1$, orient $\{l_i, r_j\}$ from l_i to r_j.

(2) For all i, j such that $(v_i, v_j) \in O$, orient $\{r_i, l_j\}$ from r_i to l_j.

The desired set of intervals exists iff this partial orientation is acyclic, because in this case, the partial orientation can be extended to a linear ordering of $Q(G)$ from which we can obtain the desired intervals.

3. A second description of chronological orderings

It may have become clear in the last section that to completely describe a chronological ordering of a graph, all that is needed is the linear order of the subset $\{r_1, \dots, r_n\}$ of P and the linear order of the subset $\{l_1, \dots, l_n\}$. Indeed, let T be a linear ordering of P that is a chronological ordering of a graph G. If we restrict T to the subgraph of $Q(G)$ induced by $V_R = \{r_1, \dots, r_n\}$ and to the subgraph of $Q(G)$ induced by $V_L = \{l_1, \dots, l_n\}$, we obtain linear orderings T_R of V_R and T_L of V_L. Conversely, given T_R and T_L, we can reconstruct T by extending T_R and T_L to an orientation of $Q(G)$ in the following manner: For all $\{l_i, r_j\}$,

(1) if $i=j$ or $\{v_i, v_j\} \in E$, then orient $\{l_i, r_j\}$ from l_i to r_j,

(2) if $\{v_i, v_j\} \notin E$ and $(r_i, r_j) \in T_R$, then orient $\{l_i, r_j\}$ from l_i to r_j. In the case where $(r_j, r_i) \in T_R$, orient $\{l_i, r_j\}$ from r_j to l_i.

We could, of course, have used (l_i, l_j) or $(l_j, l_i) \in T_L$ instead of (r_i, r_j) or $(r_j, r_i) \in T_R$ in part (2).

This raises questions concerning T_R and T_L similar to those asked of T in Section 2. The question we discuss here is: Given an interval graph G, which linear orderings T_R and T_L of V_R and V_L, respectively, give chronological orderings of G? The following theorem gives two conditions on T_R and T_L necessary and sufficient for them to give a chronological ordering of G. In fact, the theorem is slightly stronger. It actually characterizes interval graphs, in that G is an interval graph iff there exist linear orderings T_R and T_L with the two stated properties.

For each vertex $v \in V$, we define the *closed neighborhood*

$$N(v) = \{w \in V: \{v, w\} \in E \text{ or } v = w\}.$$

Theorem 2. *Let $G=(V,E)$ be a graph and T_R and T_L linear orderings of V_R and V_L*

respectively. Then G is an interval graph for which T_R and T_L give a chronological ordering iff T_R and T_L have the following two properties: For all i, j, k,
(a) *if $(r_i, r_j) \in T_R$ and $v_k \in N(v_i) - N(v_j)$, then $(l_k, l_j) \in T_L$, and*
(b) *if $(l_i, l_j) \in T_L$ and $v_k \in N(v_j) - N(v_i)$, then $(r_i, r_k) \in T_R$.*

Proof. The properties (a) and (b) are easily seen to be necessary by considering what they say about interval representations of G. To show that they are also sufficient, assume that T_R and T_L have these two properties. Now construct an orientation T of $Q(G)$ by extending T_R and T_L as described in (1) and (2) in the first paragraph of this section. To complete the proof, we need only show that T is a transitive orientation of $Q(G)$.

We use in the proof the fact that

$$(*) \qquad \text{if } \{v_i, v_j\} \notin E, \text{ then } (l_i, l_j) \in T_L \text{ iff } (r_i, r_j) \in T_R \text{ iff } (r_i, l_j) \in T \text{ iff } (l_i, r_j) \in T.$$

The first equivalence can be demonstrated by applying $k=i$ and $k=j$ in properties (a) and (b), respectively, and the rest are true by the definition of T.

By the linearity of T_R and T_L and by symmetry, we can obtain the transitivity of T by proving the following three statements:
(i) If $(l_i, l_j), (l_j, r_k) \in T$, then $(l_i, r_k) \in T$.
(ii) If $(r_i, r_j), (r_j, l_k) \in T$, then $(r_i, l_k) \in T$.
(iii) If $(l_i, r_j), (r_j, l_k) \in T$, then $(l_i, l_k) \in T$.

(i). Let $(l_i, l_j), (l_j, r_k) \in T$. We assume $(r_k, l_i) \in T$ and obtain a contradiction. If $(r_k, l_i) \in T$, then from the definition of T, we must have $\{v_k, v_i\} \notin E$ and therefore $(r_k, r_i), (l_k, l_i) \in T$, by $(*)$. By the transitivity of T_L, this implies that $(l_k, l_j) \in T$. Since we now have both $(l_k, l_j), (l_j, r_k) \in T$, then by $(*)$, we must have $\{v_j, v_k\} \in E$. However, if $\{v_j, v_k\} \in E$, then property (b) implies that $(r_i, r_k) \in T$ (contradiction).

(ii). Let $(r_i, r_j), (r_j, l_k) \in T$. Then by the definition of T, $\{v_j, v_k\} \notin E$, and so, by $(*)$, $(l_j, l_k), (r_j, r_k) \in T$. By the transitivity of T_R, this implies that $(r_i, r_k) \in T$. Now, if $\{v_i, v_k\} \in E$, then by property (a), $(l_k, l_j) \in T$, contradicting the fact that $(l_j, l_k) \in T$. Therefore, we must have $\{v_i, v_k\} \notin E$, and hence by $(*)$ and the fact that $(r_i, r_k) \in T$, we obtain the desired result that $(r_i, l_k) \in T$.

(iii). Let $(l_i, r_j), (r_j, l_k) \in T$. Then by the definition of T, $\{v_j, v_k\} \notin E$ and so by $(*)$, $(r_j, r_k), (l_j, l_k) \in T$. Now, if $\{v_i, v_j\} \notin E$, then by $(*)$, $(l_i, l_j) \in T$, and hence $(l_i, l_k) \in T$ as desired by the transitivity of T_L. But, if $\{v_i, v_j\} \in E$, we also obtain $(l_i, l_k) \in T$, since, if $(l_k, l_i) \in T$ instead, then property (b) would imply that $(r_k, r_j) \in T$, which gives us a contradiction.

This completes the proof of the theorem. $\square$

4. A third description of chronological orderings

There is a third way of describing chronological orderings of interval graphs. Let $\{I_1, \ldots, I_n\}$ denote an interval representation of $G=(V, E)$ with distinct endpoints.

Notice that every pair of intervals, I_i and I_j, in the representation are related in one of the following ways:

(1) I_i is contained in I_j,

(2) I_i overlaps I_j on the left, or

(3) I_i is followed by I_j.

Of course, the roles of i and j could be reversed as well. This motivates the following construction.

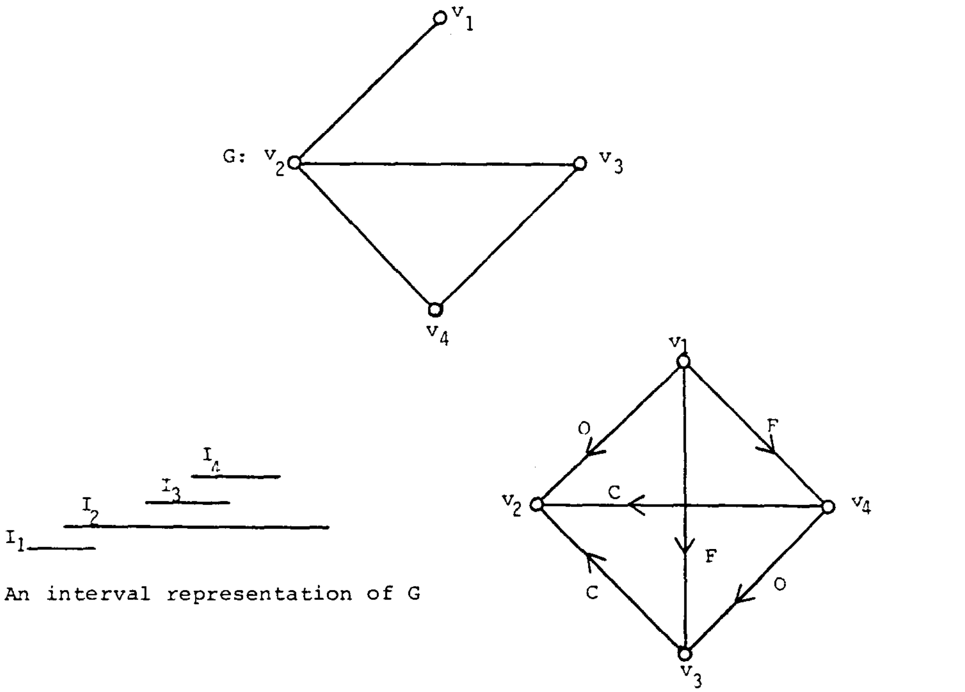
Let $D=(V;\, C, O, F)$ denote a tournament (an oriented complete graph) whose arc set is partitioned into three sets C, O, F (to represent containment, overlap, and following, respectively). Given a chronological ordering T of G, we can construct such a tournament D as follows: Let

$$C=\{(v_i, v_j) \colon l_j<l_i<r_i<r_j\},$$

$$O=\{(v_i, v_j) \colon l_i<l_j<r_i<r_j\},$$

$$F=\{(v_i, v_j) \colon l_i<r_i<l_j<r_j\}.$$

In this way, every chronological ordering of G can be uniquely described by a tournament $D=(V;\, C, O, F)$. For example, see Fig. 4. Note that the edges of D are directed in a way that corresponds to the orientation T_{R}, i.e., $(r_i, r_j) \in T_{\mathrm{R}}$ iff



An interval representation of G

The corresponding tournament D

Fig. 4

$(v_i, v_j) \in C \cup O \cup F$. Similarly, $(l_i, l_j) \in T_L$ iff $(v_i, v_j) \in C^{-1} \cup O \cup F$, where $(v_i, v_j) \in C^{-1}$ iff $(v_j, v_i) \in C$. These facts are used in the following theorem which states exactly which tournaments $D = (V; C, O, F)$ correspond to chronological orderings of G. Actually, like Theorem 2, this theorem characterizes interval graphs, in that G is an interval graph iff there exists a tournament D with the properties given.

For two sets A and B of arcs of a digraph, we let

$$AB = \{(x, z): (x, y) \in A \text{ and } (y, z) \in B \text{ for some vertex } y\}.$$

Thus an orientation A is transitive iff $A^2 \subset A$.

Theorem 3. *Let $G = (V, E)$ be a graph and $D = (V; C, O, F)$ be a tournament. Then G is an interval graph which has a chronological ordering corresponding to D iff D has the following five properties:*

(a) *The underlying set of undirected edges of $C \cup O$ is E.*
(b) $C^2 \subset C$.
(c) *For all $(v_i, v_j) \in C$, $N(v_i) \subset N(v_j)$.*
(d) $(O \cup F)^2 \subset O \cup F$.
(e) $OF \cup FO \cup FF \subset F$.

Proof. The necessity of these conditions can easily be seen when one considers what they say about an interval representation of G. Therefore, assume now that D has the five properties (a)–(e). We show that these properties are sufficient by using C, O, and F to construct linear orderings T_R and T_L of V_R and V_L, respectively, that have the two properties in Theorem 2. By the construction of T_R and T_L, it can easily be seen that the chronological ordering corresponding to T_R and T_L also corresponds to D.

Define T_R and T_L as follows: For all i, j,

$$(r_i, r_j) \in T_R \quad \text{iff} \quad (v_i, v_j) \in C \cup O \cup F,$$

$$(l_i, l_j) \in T_L \quad \text{iff} \quad (v_i, v_j) \in C^{-1} \cup O \cup F.$$

We need to show that T_R and T_L are linear orderings. However, C (and hence also C^{-1}) is transitive, as is $O \cup F$. Thus $C \cup O \cup F$ (and $C^{-1} \cup O \cup F$) is just a union of two complementary transitive orientations, which must form a linear ordering as the following argument shows. Assume that $C \cup O \cup F$ is not a linear ordering. Then it contains a cycle. By the transitivity of C and $O \cup F$, this cycle can be reduced to one whose arcs alternate between C and $O \cup F$. By considering the chords of this cycle, it can be further reduced to a cycle of length 4. But now, regardless of the orientations of the diagonals of this cycle or whether they are in C or $O \cup F$, we obtain a contradiction. Therefore $C \cup O \cup F$ and $C^{-1} \cup O \cup F$ must be linear orderings of V and hence T_R and T_L are linear orderings of V_R and V_L respectively.

To show that T_R and T_L satisfy condition (a) in Theorem 2, assume that

$(r_i, r_j) \in T_R$ and $v_k \in N(v_i) - N(v_j)$. We need to show that $(l_k, l_j) \in T_L$, which is equivalent to showing that $(v_k, v_j) \in C^{-1} \cup O \cup F$. By property (c) above, we cannot have $(v_i, v_j) \in C$. Hence $(v_i, v_j) \in O \cup F$. Similarly, since $\{v_j, v_k\} \notin E$, property (a) above implies that either $(v_j, v_k) \in F$ or $(v_k, v_j) \in F$. However, if $(v_j, v_k) \in F$, then property (e) above tells us that $(v_i, v_k) \in F$, contradicting property (a) and the fact that $\{v_i, v_k\} \in E$. Therefore, we must have $(v_k, v_j) \in F$, which proves that $(l_k, l_j) \in T_L$.

The proof of the second condition in Theorem 2 is symmetric. □

This theorem arms us with new techniques for solving problems concerning chronological orderings. For example, let us return to our problem in archaeological seriation. Suppose that, from the information obtained from the graves, the archaeologists could determine exactly which styles of pottery had intervals of use that were contained in the intervals of use of other styles. Then the archaeologists could partition the edges of their interval graph into two sets, one set consisting of those edges that indicate containment and the other consisting of those edges that indicate overlap. Furthermore, in this case, the edges indicating containment could be oriented to indicate which interval is contained in which. The archaeologists are then presented with the problem of finding chronological orderings of the interval graph consistent with this data. For example, see Fig. 5.

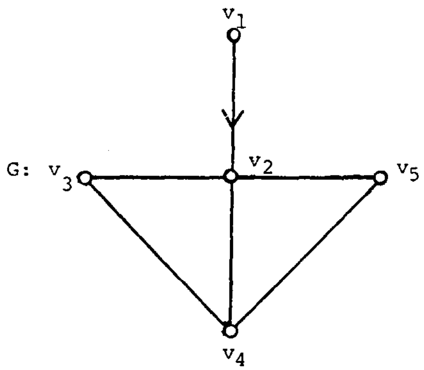


Fig. 5. Suppose that the only edge of G indicating containment is the oriented edge (v_1, v_2). Does G have an interval representation $\{I_1, \ldots, I_5\}$ with distinct endpoints in which I_1 is contained in I_2 but no other interval is properly contained in another?

This problem can be stated more generally. Let K denote the set of edges in $\bar{G}$, i.e., $\{v_i, v_j\} \in K$ iff $\{v_i, v_j\} \notin E$. Then a more general version of our problem becomes: Given a graph $G = (V, E)$ whose edge set E is partitioned into two subsets M and H, and given partial orientations of M, H, and K, can these partial orientations be extended to orientations C, O, and F, of M, H, and K, respectively, so that the resulting tournament $D = (V; C, O, F)$ corresponds to a chronological ordering of G?

Using Theorem 3, this question can be answered. Furthermore, this theorem provides means for determining this answer algorithmically in a fairly efficient manner. From the properties listed in Theorem 3, it can be seen that the orientation C is independent of the orientations O and F. Therefore, we can split our problem into two parts:

(1) Determine whether the partial orientation of M can be extended to a transitive orientation C of M that has the property that $N(v_i) \subset N(v_j)$, for all $(v_i, v_j) \in C$.

(2) Determine whether the partial orientations of H and K can be extended to a transitive orientation $O \cup F$ of $H \cup K$ with the property that $OF \cup FO \cup FF \subset F$.

We can apply the methods in [3], [5] or [6] as mentioned in section 2 above to obtain an algorithm for these problems. In part (1) of our problem, we can ensure that our extension will satisfy the requirement that $N(v_i) \subset N(v_j)$ for all $(v_i, v_j) \in C$ by, as a preliminary step, extending the partial orientation of M as follows: For all $\{v_i, v_j\} \in M$ such that $N(v_i) \subsetneq N(v_j)$, orient $\{v_i, v_j\}$ from v_i to v_j. We then apply the methods mentioned above on this extended partial orientation to try to obtain a transitive orientation.

A more complex modification of these methods is needed to obtain the extension desired in part (2) of our problem, due to our extra requirement that $OF \cup FO \cup FF \subset F$. A modification of the algorithm in [5] or [6] that produces such an extension (if possible) is discussed in [13]. Our extra requirement has the following effect. If we have a triangle two of whose edges are in K and one is in H, then the requirement that $F^2 \subset F$ forces the two edges in K to be oriented both toward or both away from their common vertex (see Fig. 6(a), (b)). If we have a triangle two of whose edges are in H and one is in K, then the requirement that $OF \cup FO \subset F$ forces the triangle to be oriented as in Fig. 6(c). These extra forcings require us to consider larger implication classes that those used in [5] or [6], but, with this modification, the rest of their methods can be applied.

The efficiency of the resulting algorithm appears to be no better than $O(n^3)$. This is because of the need to find two complementary transitive orientations.

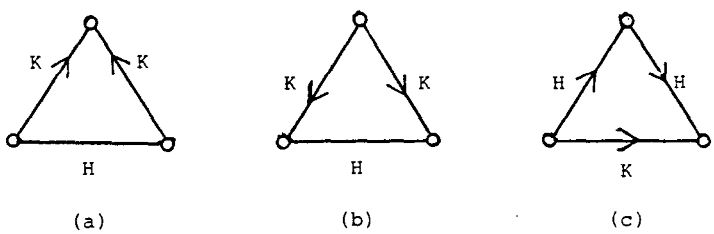


Fig. 6.

5. An application to interval counts

We can apply the material in the last section to the subject of interval counts of interval graphs. The *interval count* of an interval graph G is the minimum number of different lengths of intervals needed in constructing an interval representation of G. For example, the interval count of G is 1 iff G is a unit interval graph. For some recent results concerning interval counts, see [10]. The following theorem characterizes those graphs of interval count 2 or less for which one of the two lengths of the intervals is 0. That is, it characterizes those graphs that have representations containing only points and unit intervals.

If $S \subset V$, let $G-S$ denote the subgraph of G induced by $V-S$. Also, call a vertex v *simplicial* if $N(v)$ induces a complete subgraph of G. Clearly, if a graph G has a representation containing only points and unit intervals, then those vertices represented by points must be simplicial. Conversely, all simplicial vertices can be represented by points if desired. Furthermore, if S denotes the set of simplicial vertices of G, then the graph $G-S$ must be a unit interval graph. However, the conditions that G is an interval graph and $G-S$ is a unit interval graph are not sufficient for our problem, as the graph in Fig. 7 shows. Theorem 4 provides us with necessary and sufficient conditions.

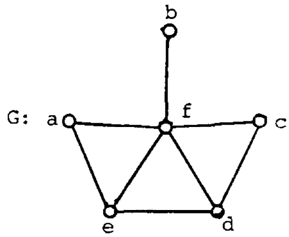


Fig. 7. Vertices a, b, and c are simplicial. G is an interval graph and $G-\{a,b,c\}$ is a unit interval graph, but G has no representation containing only points and unit intervals.

Theorem 4. *Let $G=(V,E)$ be a graph and let S denote the set of simplicial vertices of G. Then G is an interval graph that has an interval representation containing only points and unit intervals iff there exist orientations O of $G-S$ and F of $\bar{G}$ such that*

(a) $(O \cup F)^2 \subset O \cup F$, *and*

(b) $OF \cup FO \cup FF \subset F$.

Proof. We only prove the theorem in the case where no two simplicial vertices are adjacent. The general case follows readily from this because two or more adjacent simplicial vertices can be represented by the same point or by nested intervals.

Suppose G has a representation containing only points and unit intervals. We make minor modifications of this representation to obtain a representation with distinct endpoints. First translate, if necessary, some of the points and unit intervals to obtain distinct endpoints for all of the unit intervals. This can easily be done without changing their intersection properties. Now, shrink or expand the intervals representing the simplicial vertices to small intervals contained in the interior of every other interval with which they intersect. This can be done since no two simplicial vertices are adjacent. The resulting representation of G gives a chronological ordering of G whose corresponding tournament $D=(V; C, O, F)$ satisfies the condition that $(x, y) \in C$ iff $\{x, y\} \in E$ and $x \in S$. Thus by Theorem 3, we have proven the necessity of the existence of the orientations O and F.

Conversely, suppose that $G-S$ and $\bar{G}$ have orientations O and F with the properties described in the theorem. Define an orientation C on those edges of G not oriented in O by letting $(x, y) \in C$ iff $\{x, y\} \in E$ and $x \in S$. Clearly $C^2 \subset C$ and $N(x) \subset N(y)$ if $(x, y) \in C$. Therefore, the tournament $D=(V; C, O, F)$ has all the properties required in Theorem 3 for G to have an interval representation corresponding to D. Consider now such a representation of G.

Since all the neighbors of each $v \in S$ are adjacent, no intersections are created or obliterated if the interval corresponding to each such v is shrunk to a point. Also we can expand or shrink each interval that represents a vertex in $G-S$ into a unit interval. This can be done without changing any of the intersection properties because no such interval is properly contained in another. In this way, we obtain the desired representation. □

We remark that this can be checked algorithmically in $O(n^3)$ steps using the methods discussed in the previous section.

Acknowledgments

This research was supported in part by a grant from the Office of Naval Research. The work forms a part of the author's doctoral dissertation written under the supervision of Professor Victor Klee at the University of Washington.

The author would like to thank the referees of this paper for their many helpful suggestions.

References

[1] S. Benzer, The fine structure of the gene, Sci. Amer. 206 (1962) 70-84.
[2] S. Benzer, On the topology of the genetic fine structure, Proc. Nat. Acad. Sci. USA 45 (1959) 1607-1620.
[3] F. Gavril, A recognition algorithm for the intersection graphs of paths in trees, Discrete Math. 23 (1978) 211-227.
[4] P. Gilmore and A. Hoffman, A characterization of comparability graphs and of interval graphs, Canad. J. Math. 16 (1964) 539-548.
[5] M. Golumbic, Algorithmic Graph Theory and Perfect Graphs (Academic Press, New York, 1980).
[6] M. Golumbic, Comparability graphs and a new matroid, J. Combin. Theory B22 (1977) 68-90.
[7] M. Golumbic, The complexity of comparability graph recognition and coloring, Computing 18 (1977) 199-208.
[8] D. Kendall, Incidence matrices, interval graphs and seriation in archaeology, Pacific J. Math. 28 (1969) 565-570.
[9] D. Kendall, Some problems and methods in statistical archaeology, World Archaeology 1 (1969) 68-76.

[10] R. Leibowitz, Interval counts and threshold graphs, Ph.D. Thesis, Rutgers University, New Brunswick, NJ (1978).
[11] F. Roberts, Discrete Mathematical Models, With Applications to Social, Biological and Environmental Problems (Prentice-Hall, Englewood Cliffs NJ, 1976).
[12] F. Roberts, Graph Theory and Its Applications to Problems of Society, NFS-CBMS Monograph No. 29 (SIAM Publications, Philadelphia, 1978).
[13] D. Skrien, Interval graphs, chronological orderings, and related matters, Ph.D. Thesis, University of Washington, Seattle, WA (1980).